$G_X(s)$: prob g.f. for r.v. X evaluated at s defined as $\mathbb{E}s^X$ (discrete r.v. X)

motivation: ①: g.f. 1-to-1 correspondence with dist

②: Deal with sum of independent r.v.

X, Y r.v. indep.

$$\mathbb{E}s^{X+Y} = \mathbb{E}\underbrace{s^X}\cdot\underbrace{s^Y} = \mathbb{E}s^X \cdot \mathbb{E}s^Y$$

e.g. (5.2.3)

(a): $X|_Y \sim \mathcal{P}(Y)$, $Y \sim \mathcal{P}(\mu)$, find $G_{X+Y}(s)$.

Pf: $G_{X+Y}(s) = \mathbb{E}s^{X+Y} = \mathbb{E}\left[\mathbb{E}(s^{X+Y}|Y)\right]$

first calculate

$$\mathbb{E}(s^{X+Y}|Y=y) = \mathbb{E}(s^{X+y}|Y=y) = s^y\cdot\mathbb{E}(s^X|Y=y) = s^y\cdot\sum_{k=0}^{\infty} s^k\cdot\frac{y^k}{k!}e^{-y} = e^{sy-y}\cdot s^y$$

So: $\mathbb{E}(s^{X+Y} | Y) = s^Y \cdot e^{Y(s-1)}$

$$G_{X+Y}(s) = \mathbb{E}\left[s^Y \cdot e^{Y(s-1)}\right]$$

$$= \sum_{k=0}^{\infty} s^k \cdot e^{k(s-1)} \cdot \frac{\mu^k}{k!} e^{-\mu}$$

$$= \sum_{k=0}^{\infty} \frac{\left[s \cdot e^{s-1} \cdot \mu\right]^k}{k!} e^{-\mu}$$

$$= \underline{\underline{e^{s \cdot e^{s-1} \cdot \mu - \mu}}}$$

compounding of r.v.

(b): $X_1, X_2, \cdots$ i.i.d. r.v. $f(k) = \frac{(1-p)^k}{k \log \frac{1}{p}}$ $(k \geq 1)$ where $p \in (0,1)$. If N indep of $\{X_i\}$ and $N \sim \mathcal{P}(\mu)$, show that $Y = \sum_{i=1}^{N} X_i$ has NB dist.

__Pf__: $G_Y(s) = \mathbb{E} s^Y = \mathbb{E}\, s^{\sum_{i=1}^{N} X_i} = \mathbb{E}\left(\mathbb{E}\left[s^{\sum_{i=1}^{N} X_i} \middle| N\right]\right)$

Firstly,

$$\mathbb{E}\left(s^{\sum_{i=1}^{N} X_i} \middle| N=n\right) = \mathbb{E}\left(s^{\sum_{i=1}^{n} X_i} \middle| N=n\right)$$

independence of N and $\{X_i\}$

$$= \mathbb{E}\left(s^{\sum_{i=1}^{n} X_i}\right) \overset{\{X_i\} \text{ i.i.d.}}{=} \left(\mathbb{E} s^{X_1}\right)^n$$

$$= \left[\sum_{k=1}^{\infty} s^k \cdot \frac{(1-p)^k}{k \cdot \log \frac{1}{p}}\right]^n$$

$$= \left(\frac{1}{\log \frac{1}{p}} \cdot \sum_{k=1}^{\infty} \frac{[s(1-p)]^k}{k} \right)^n$$

Consider $\frac{d}{dq}\left(\sum_{k=1}^{\infty} \frac{q^k}{k}\right) \overset{\text{just}\sim}{=} \sum_{k=1}^{\infty} \frac{d}{dq} \frac{q^k}{k} = \sum_{k=1}^{\infty} q^{k-1}$

$$= \frac{1}{1-q}$$

$$\sum_{k=1}^{\infty} \frac{q^k}{k} = -\log(1-q) + C$$

if set $q=0$, $LHS=0$, $RHS=C$, so $\underline{C=0}$

get: $\sum_{k=1}^{\infty} \frac{q^k}{k} = -\log(1-q)$.

justification of interchange given by the uniform convergence of series on any compact subset of the convergence domain.

$$= \left(\frac{-\log[1-s(1-p)]}{\log \frac{1}{p}} \right)^n = \left(\frac{\log[1-s(1-p)]}{\log p} \right)^n$$

So: $G_Y(s) = \mathbb{E}\left(\frac{\log[1-s(1-p)]}{\log p} \right)^N$

$$= \sum_{k=0}^{\infty} \left(\text{———} \right)^k \cdot \frac{\mu^k}{k!} e^{-\mu}$$

$$= e^{\mu \cdot \left(\frac{\log[1-s(1-p)]}{\log p} - 1 \right)} = e^{\mu \cdot \frac{\log \frac{1-s(1-p)}{p}}{\log p}}$$

$$= \left(\frac{1-s(1-p)}{p}\right)^{\frac{\mu}{\log p}} = \left[\frac{p}{1-s(1-p)}\right]^{-\frac{\mu}{\log p}}.$$

g.f. of $NB(r,p)$, $q(k) = \binom{r+k-1}{k} p^r (1-p)^k$

$$\mathbb{E}\, s^{Z} = \sum_{k=0}^{\infty} s^k \cdot \binom{r+k-1}{k} p^r (1-p)^k$$

$$= p^r \cdot \sum_{k=0}^{\infty} [s(1-p)]^k \cdot \binom{r+k-1}{k}$$

$$\binom{r+k-1}{k} = \frac{(r+k-1)!}{k!\,(r-1)!} = \frac{(r+k-1)\cdot(r+k-2)\cdot\cdots\cdot r}{k!} = (-1)^k \cdot \frac{(-r)(-r-1)\cdot\cdots\cdot(-r-k+1)}{k!} = (-1)^k \cdot \binom{-r}{k}$$

$$= p^r \cdot \sum_{k=0}^{\infty} \binom{-r}{k} \cdot [-s(1-p)]^k$$

$$= [1-s(1-p)]^{-r} \cdot p^r$$

$$= \left[\frac{1-s(1-p)}{p}\right]^{-r}$$

$$= \left[\frac{p}{1-s(1-p)}\right]^{r}$$

conclude: $Y \sim NB\left(r = -\frac{\mu}{\log p},\ p = p\right).$

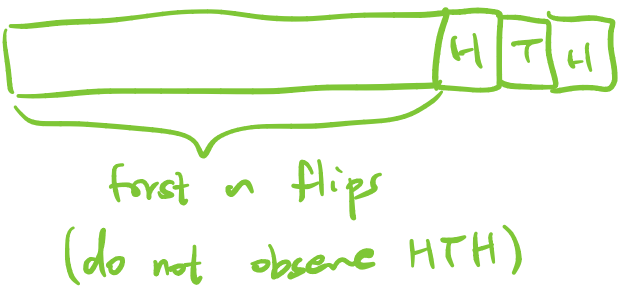
e.g.: (5.2.6) $\begin{cases} H \to p \\ T \to q \end{cases}$ $(p+q=1)$

$X = \#$ of flips until HTH, g.f. of X

Trick: Consider event

$A \triangleq \{X>n, \text{ followed by HTH}\}$

motivation: purposefully add stopping criterion "HTH" at the end to make sure the value of X under this event is restricted.



$$\mathbb{P}(A) \overset{\text{indep}}{=\!=} \mathbb{P}(X>n) \cdot \mathbb{P}(HTH) = p^2 q\, \mathbb{P}(X>n)$$

notice under event A, X only take values $X = n+1$ or $n+3$

$$\mathbb{P}(A) \underset{\text{values of } X}{\overset{\text{discuss}}{=\!=}} \mathbb{P}(A, X=n+1) + \mathbb{P}(A, X=n+3)$$

$$= \mathbb{P}(X=n+1) \cdot qp + \mathbb{P}(X=n+3)$$

$$p^2 q\, \mathbb{P}(X>n) = pq\, \mathbb{P}(X=n+1) + \mathbb{P}(X=n+3)$$

multiply both sides by s^{n+3} and sum w.r.t. n

$$p^2 q \boxed{\sum_{n=0}^{\infty} \mathbb{P}(X>n)\cdot s^{n+3}} = pq \underbrace{\sum_{n=0}^{\infty} \mathbb{P}(X=n+1)\, s^{n+3}}_{} \rightarrow s^2 \cdot G_X(s)$$

$$\| \qquad\qquad + \underbrace{\sum_{n=0}^{\infty} \mathbb{P}(X=n+3)\cdot s^{n+3}}_{G_X(s)}$$

$$s^3 \cdot \sum_{n=0}^{\infty} \mathbb{P}(X>n)\cdot s^n$$

$$\sum_{n=0}^{\infty} \mathbb{P}(X>n)\cdot s^n = \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} \mathbb{P}(X=k)\cdot s^n$$

$$\underset{\substack{\text{interchange} \\ \text{sum}}}{\overset{\text{Fubini}}{=}} \sum_{k=1}^{\infty} \sum_{n=0}^{k-1} \mathbb{P}(X=k)\cdot s^n$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(X=k)\cdot \sum_{n=0}^{k-1} s^n$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(X=k)\cdot \frac{1-s^k}{1-s}$$

$$= \frac{1 - G_X(s)}{1-s}$$

$$p^2 q s^3 \frac{1-G_X(s)}{1-s} = pq s^2 G_X(s) + G_X(s)$$

$$\frac{p^2 q s^3}{1-s} = \left(1 + pq s^2 + \frac{p^2 q s^3}{1-s}\right) G_X(s)$$

$$G_X(s) = \frac{\frac{p^2 q s^3}{1-s}}{1 + pqs^2 + \frac{p^2 q s^3}{1-s}}$$

$$= \frac{p^2 q s^3}{1 - s + pqs^2 - pqs^3 + p^2 q s^3}$$

$$p^2 q - pq$$
$$= pq(p-1)$$
$$= -pq^2$$

$$= \frac{p^2 q s^3}{1 - s + pqs^2 - pq^2 s^3}$$

Sanity Check:

If $X < \infty$ a.s., then $G_X(1) = 1$

$G_X(1) = \sum_{n=0}^{\infty} \mathbb{P}(X=n) = 1$ if $X < \infty$ a.s.

in this case,

$$G_X(1) = \frac{p^2 q}{pq - pq^2} = \frac{p}{1-q} = \frac{p}{p} = 1.$$

Y = # of flips until either HTH or THT happens

find g.f. of Y.

Z = # of flips —— THT happens,

$Y = \min\{X, Z\} \Rightarrow$ either $X = Y$ or $Y = Z$.

first n flips

(no HTH and no THT)

$A_1 = \{Y > n,$ followed by HTH$\}$

$\Downarrow$

$Y = n+1, n+2, n+3$

first n flips

(no HTH and no THT)

$A_2 = \{Y > n,$ followed by THT$\}$

$\Downarrow$

$Y = n+1, n+2, n+3$

$$\mathbb{P}(A_1) \overset{\text{indep}}{=} \mathbb{P}(Y > n) \cdot \mathbb{P}(HTH) = \underline{p^2 q \cdot \mathbb{P}(Y > n)}$$

discuss value of Y $\Big\|$

$$\underline{\mathbb{P}(Y = X = n+1) \cdot pq + \mathbb{P}(Y = Z = n+2) \cdot p}$$

$$+ \underline{\mathbb{P}(Y = X = n+3)}$$

$$\mathbb{P}(A_2) \overset{\text{indep}}{=\!=} \mathbb{P}(Y>n)\cdot\mathbb{P}(THT) = \underline{pq^2\cdot\mathbb{P}(Y>n)}$$

discuss value of Y

$$= \underline{\mathbb{P}(Y=Z=n+1)\cdot pq + \mathbb{P}(Y=X=n+2)\cdot q}$$
$$+ \underline{\mathbb{P}(Y=Z=n+3)}$$

Fact: $\mathbb{P}(Y>n) = \mathbb{P}(X=Y,\ Y>n) + \mathbb{P}(Y=Z,\ Y>n)$

Key equations:

$$\begin{cases} p^2q\,\mathbb{P}(Y>n) = \mathbb{P}(Y=X=n+1)\,pq + \mathbb{P}(Y=Z=n+2)\,p \\ \qquad\qquad + \mathbb{P}(Y=X=n+3) \\ pq^2\,\mathbb{P}(Y>n) = \mathbb{P}(Y=Z=n+1)\,pq + \mathbb{P}(Y=X=n+2)\,q \\ \qquad\qquad + \mathbb{P}(Y=Z=n+3) \end{cases}$$

Denote: $f_Y^X(s) \triangleq \sum_{n=0}^{\infty} \mathbb{P}(X=Y,\ Y=n)\cdot s^n$

$$f_Y^Z(s) \triangleq \sum_{n=0}^{\infty} \mathbb{P}(Y=Z,\ Y=n)\cdot s^n$$

so $f_Y^X(s) + f_Y^Z(s) = \sum_{n=0}^{\infty} \mathbb{P}(Y=n)\,s^n = G_Y(s)$

only need to calculate $f_Y^X(s)$ and $f_Y^Z(s)$.

Multiply s^{n+3} and sum w.r.t. n on both sides of key equations:

$$\begin{cases} p^2qs^3\dfrac{1-G_Y(s)}{1-s} = pqs^2\cdot f_Y^X(s) + ps\, f_Y^Z(s) + f_Y^X(s) \\ pq^2s^3\dfrac{1-G_Y(s)}{1-s} = pqs^2\cdot f_Y^Z(s) + qs\, f_Y^X(s) + f_Y^Z(s) \end{cases}$$

$$\Rightarrow \begin{cases} f_Y^X(s) = \dfrac{\frac{pqs^3}{1-s}(p-pqs+p^2qs^2)}{(1+pqs^2)^2-pqs^2+\frac{pqs^3}{1-s}(1-2pqs+pqs^2)} \\ f_Y^Z(s) = \dfrac{\frac{pqs^3}{1-s}(q-pqs+pq^2s^2)}{(1+pqs^2)^2-pqs^2+\frac{pqs^3}{1-s}(1-2pqs+pqs^2)} \end{cases}$$

So:

$$G_Y(s) = \frac{pqs^3(1-2pqs+pqs^2)}{(1-s)[(1+pqs^2)^2-pqs^2]+pqs^3(1-2pqs+pqs^2)}$$

Sanity check:

$$G_Y(1) = \frac{pq(1-2pq+pq)}{pq(1-2pq+pq)} = 1$$

also get: $\begin{cases} f_Y^X(1) = \mathbb{P}(X=Y) = \dfrac{pq(p-pq+p^2q)}{pq(1-pq)} = \dfrac{p(1-q^2)}{1-pq} \\ f_Y^Z(1) = \mathbb{P}(Z=Y) = \dfrac{pq(q-pq+pq^2)}{pq(1-pq)} = \dfrac{q(1-p^2)}{1-pq} \end{cases}$

and they add up to 1.